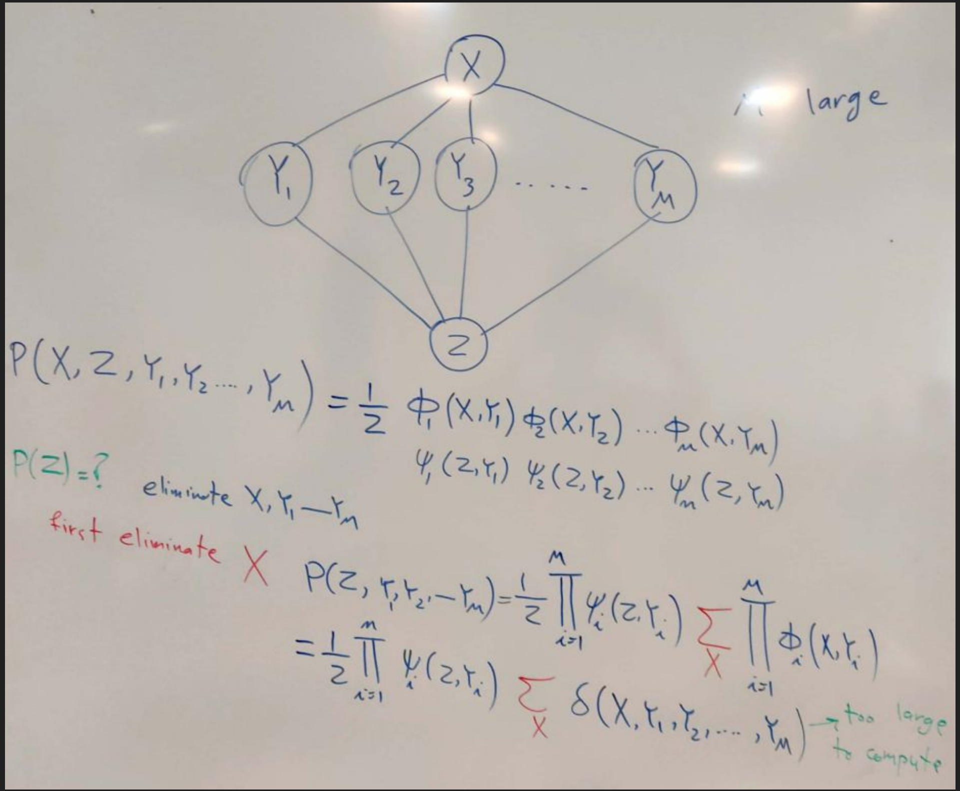
Probabilistic Graphical Models

Lectures 13

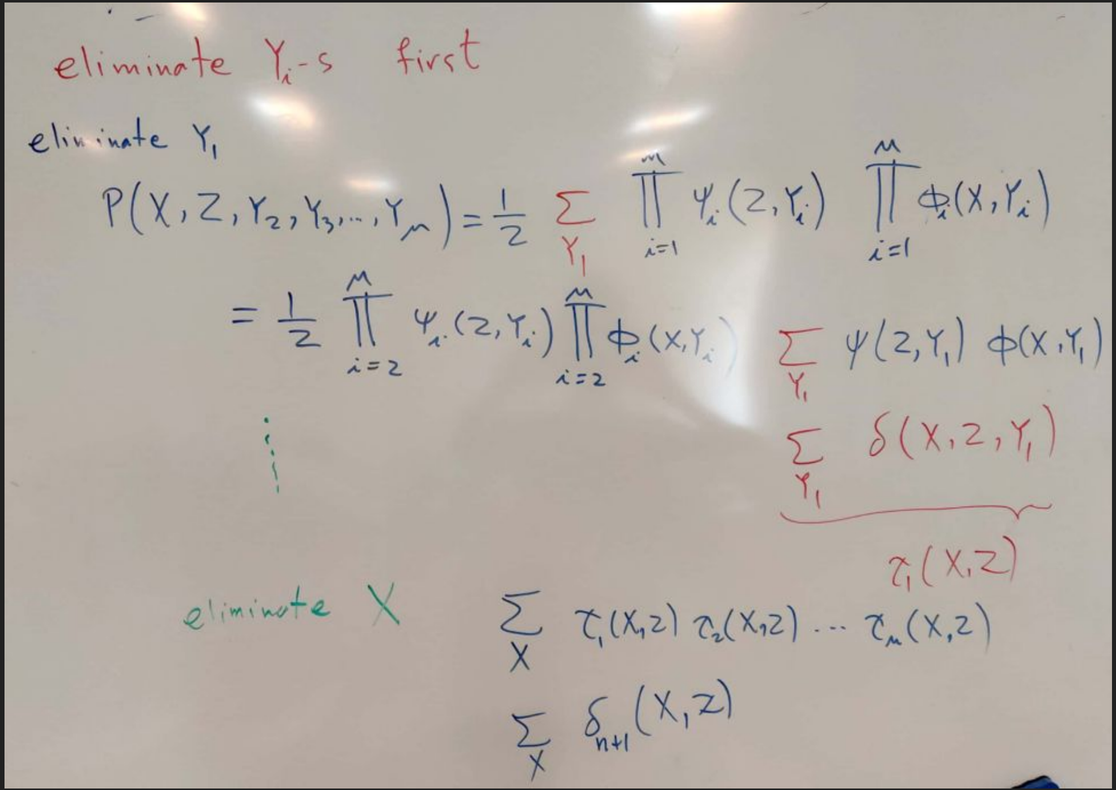
Message Passing - Junction Tree

Elimination order

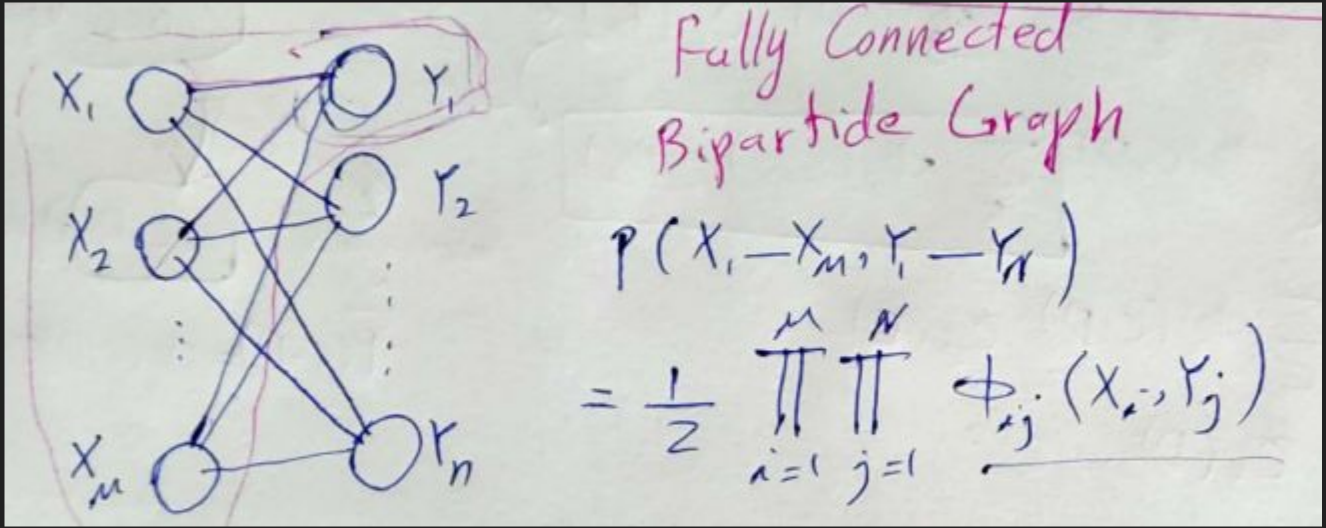


Elimination order

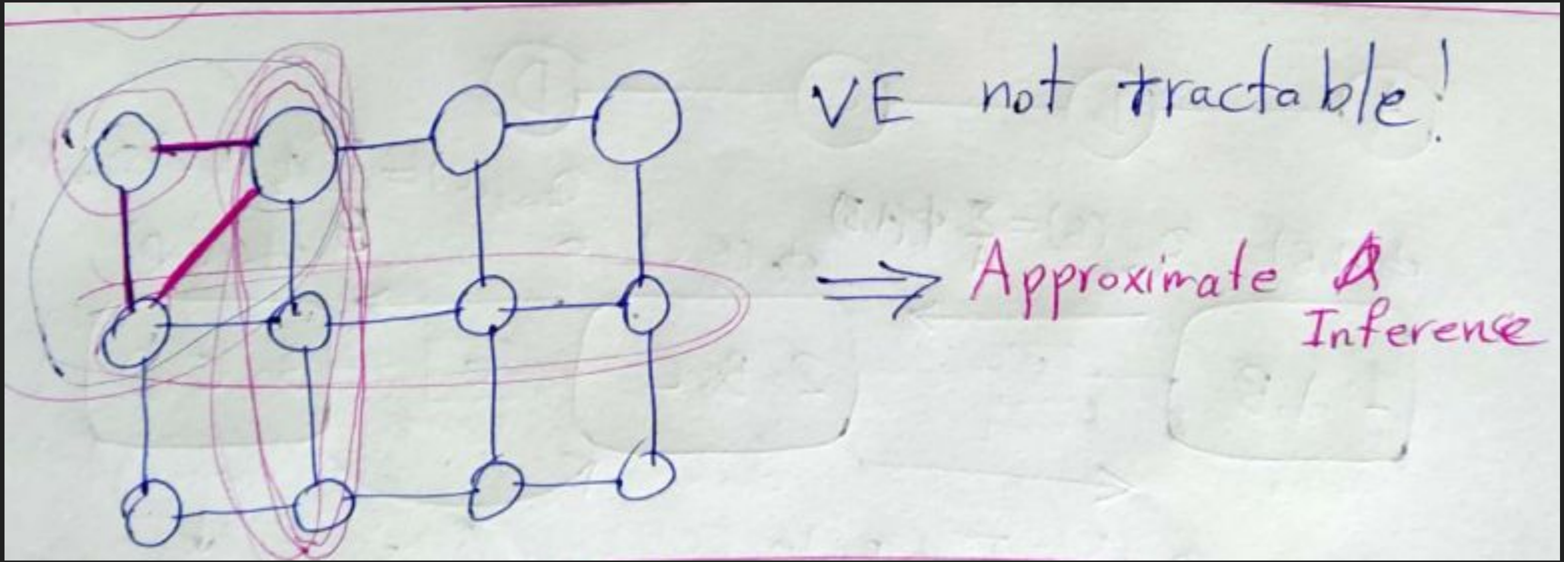
K. N. Toosi University of Technology



Variable Elimination Limitations



Variable Elimination Limitations



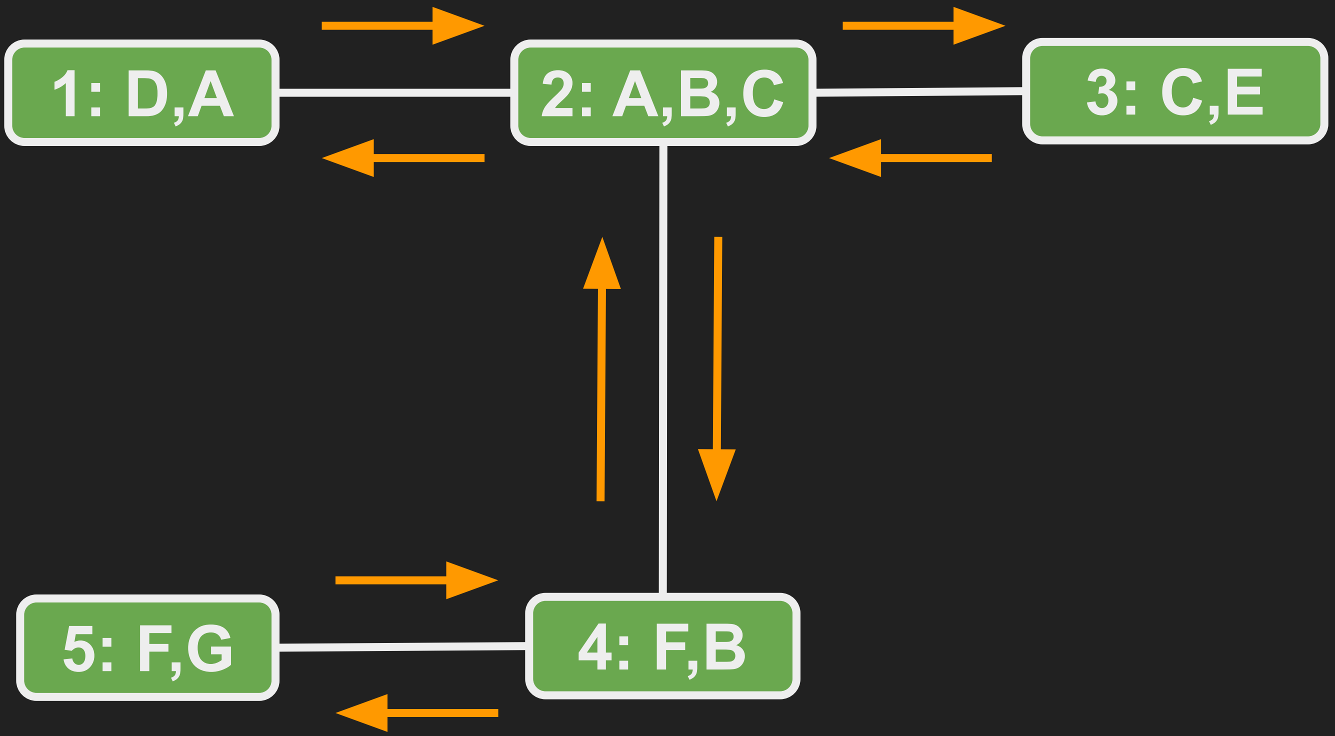
Inference

- Exact
  - Variable Elimination
  - Message Passing - Junction Tree
  - Graph-cuts
- Approximate
  - Message Passing - Loopy Belief Propagation
  - Graph-cut based
  - Sampling Based
  - Variational Inference

Variable Elimination

- Cannot compute multiple marginals using VE

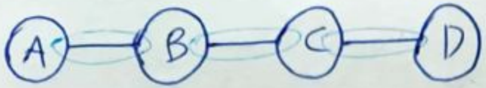
Message Passing



Message Passing

K. N. Toosi
University of Technology

17 (II)



$$P(A,B,C,D) = \frac{1}{Z}\,\phi_1(A,B)\,\phi_2(B,C)\,\phi_3(C,D)$$

$$\tilde{P}(D) = \sum_C \sum_B \sum_A \phi_1(A,B)\,\phi_2(B,C)\,\phi_3(C,D)$$

Elim. A $\displaystyle = \sum_C \sum_B \phi_2(B,C)\,\phi_3(C,D) \boxed{\sum_A \phi_1(A,B)}$

تولید

$\displaystyle = \sum_C \sum_B \phi_2(B,C)\,\phi_3(C,D)\;\tau_1(B)$

مصرف

Elim. B $\displaystyle = \sum_{C} \phi_3(C,D) \boxed{\sum_B \phi_2(B,C)\,\tau_1(B)}$

Elim. C $\displaystyle = \sum_C \phi_3(C,D)\;\tau_2(C)$

$= \tau_3(D)$

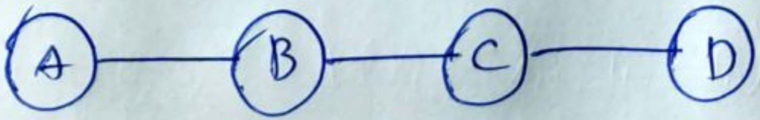
$$\begin{tikzcd}
\boxed{1: A,B}\ \phi_1(A,B) \arrow[d, "\tau_1(B)"'] \\
\boxed{2: B,C}\ \phi_2(B,C) \arrow[d, "\tau_2(C)"'] \\
\boxed{3: C,D}\ \phi_3(C,D)
\end{tikzcd}$$

messages

$$\tilde{P}(C,D) = \phi_3(C,D)\,\tau_2(C)$$

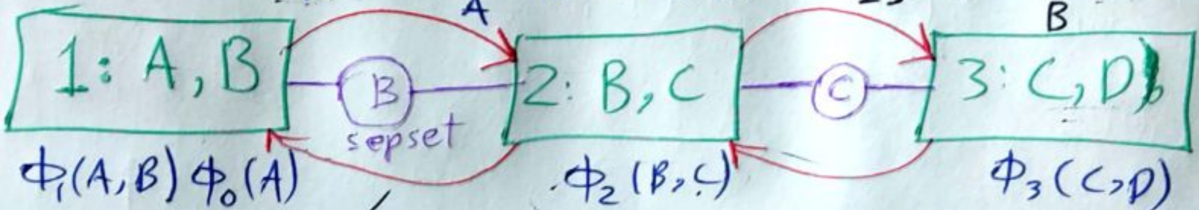
Message Passing

pgm13 II



$$P(A,B,C,D) = \phi_1(A,B)\,\phi_2(B,C)\,\phi_3(C,D)\,\phi_0(A)$$

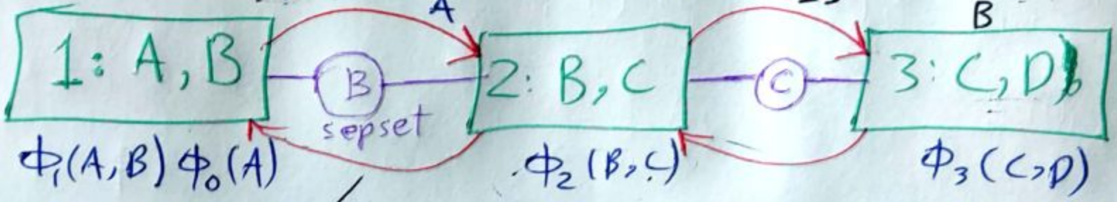
$$\tau_{12}(B) = \sum_A \phi_1(A,B)\,\phi_0(A) \qquad \tau_{23}(C) = \sum_B \tau_{12}(B)\,\phi_2(B,C)$$



$$\tau_{21}(B) = \sum_C \phi_2(B,C)\,\tau_{32}(C) \qquad \tau_{32}(C) = \sum_D \phi_3(C,D)$$

Cluster Beliefs

$$\tau_{12}(B) = \sum_A \phi_1(A,B)\,\phi_0(A) \qquad \tau_{23}(C) = \sum_B \tau_{12}(B)\,\phi_2(B,C)$$



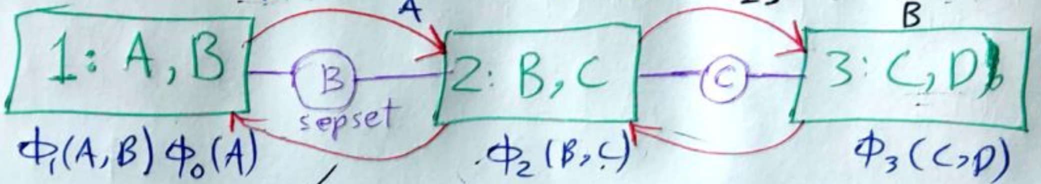
$$\tau_{21}(B) = \sum_C \phi_2(B,C)\,\tau_{32}(C) \qquad \tau_{32}(C) = \sum_D \phi_3(C,D)$$

belief

$$\begin{aligned}
\beta(B,C) &= \phi_2(B,C)\,\tau_{12}(B)\,\tau_{32}(C) \\
&= \phi_2(B,C)\Big(\sum_A \phi_1(A,B)\,\phi_0(A)\Big)\Big(\sum_D \phi_3(C,D)\Big) \\
&= \sum_A \sum_D \phi_0(A)\,\phi_1(A,B)\,\phi_2(B,C)\,\phi_3(C,D) \\
&= \sum_A \sum_D P(A,B,C,D) = P(B,C)
\end{aligned}$$

Cluster Beliefs

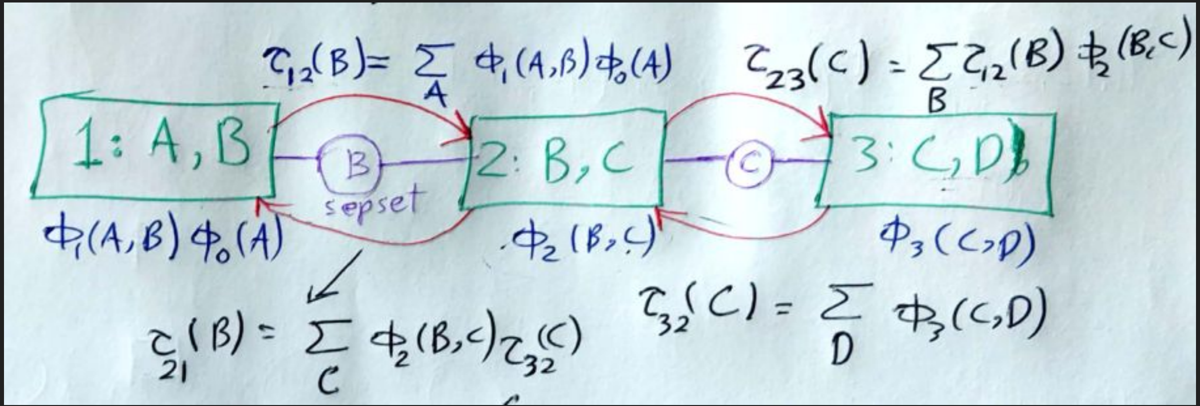
$$\tau_{12}(B) = \sum_{A} \phi_1(A,B)\,\phi_0(A) \qquad \tau_{23}(C) = \sum_{B} \tau_{12}(B)\,\phi_2(B,C)$$



$$\tau_{21}(B) = \sum_{C} \phi_2(B,C)\,\tau_{32}(C) \qquad \tau_{32}(C) = \sum_{D} \phi_3(C,D)$$

$$\begin{aligned}
\beta(C,D) &= \phi_3(C,D)\,\tau_{23}(C) = \\
&= \phi_3(C,D)\,\boxed{\sum_{B} \tau_{12}(B)\,\phi_2(B,C)} \\
&= \phi_3(C,D) \sum_{B} \boxed{\sum_{A} \phi_1(A,B)\,\phi_0(A)}\;\phi_2(B,C) \\
&= \sum_{A}\sum_{B} \phi_3(C,D)\,\phi_1(A,B)\,\phi_0(A)\,\phi_2(B,C) \\
&= \sum_{A}\sum_{B} P(A,B,C,D) = P(C,D)
\end{aligned}$$

Sepset Beliefs



pgm

$$\beta(B) = \tau_{12}(B)\,\tau_{21}(B) = \boxed{\sum_A \phi_1(A,B)\,\phi_0(A)}\;\boxed{\sum_C \phi_2(B,C)\,\tau_{32}(C)}$$

$$= \boxed{\sum_A \phi_1(A,B)\,\phi_0(A)}\;\boxed{\sum_C \phi_2(B,C)\;\boxed{\sum_D \phi_3(C,D)}}$$

$$= \sum_A \sum_C \sum_D \phi_1(A,B)\,\phi_0(A)\,\phi_2(B,C)\,\phi_3(C,D)$$

$$= \sum_A \sum_C \sum_D P(A,B,C,D) = P(B)$$

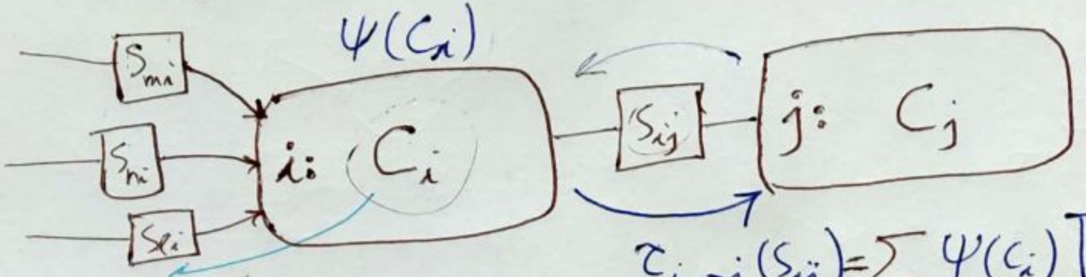
Message Passing

1: A set of clusters

2: Assign Potentials to each cluster

3: Pass messages

4: Compute the Beliefs



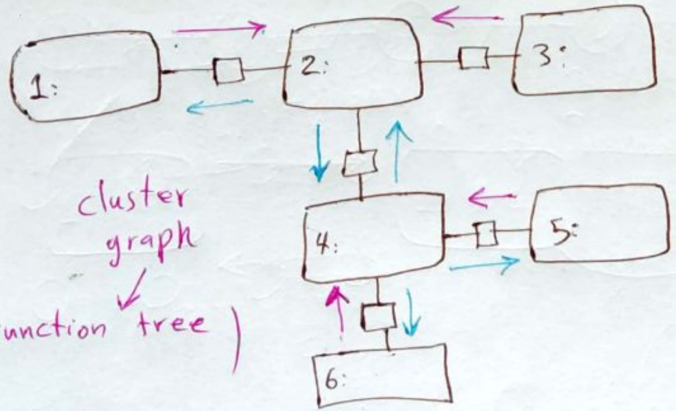
$$\tau_{i\to j}(S_{ij}) = \sum_{C_i \setminus S_{ij}} \Psi(C_i) \prod_{k \in N_i \setminus \{j\}} \tau_{k\to i}(S_{ki})$$

C_i: variable in cluster i

C_j: " " " j

$S_{ij} = C_i \cap C_j$ (sepset)

Message Passing



cluster graph
↓
(Junction tree)

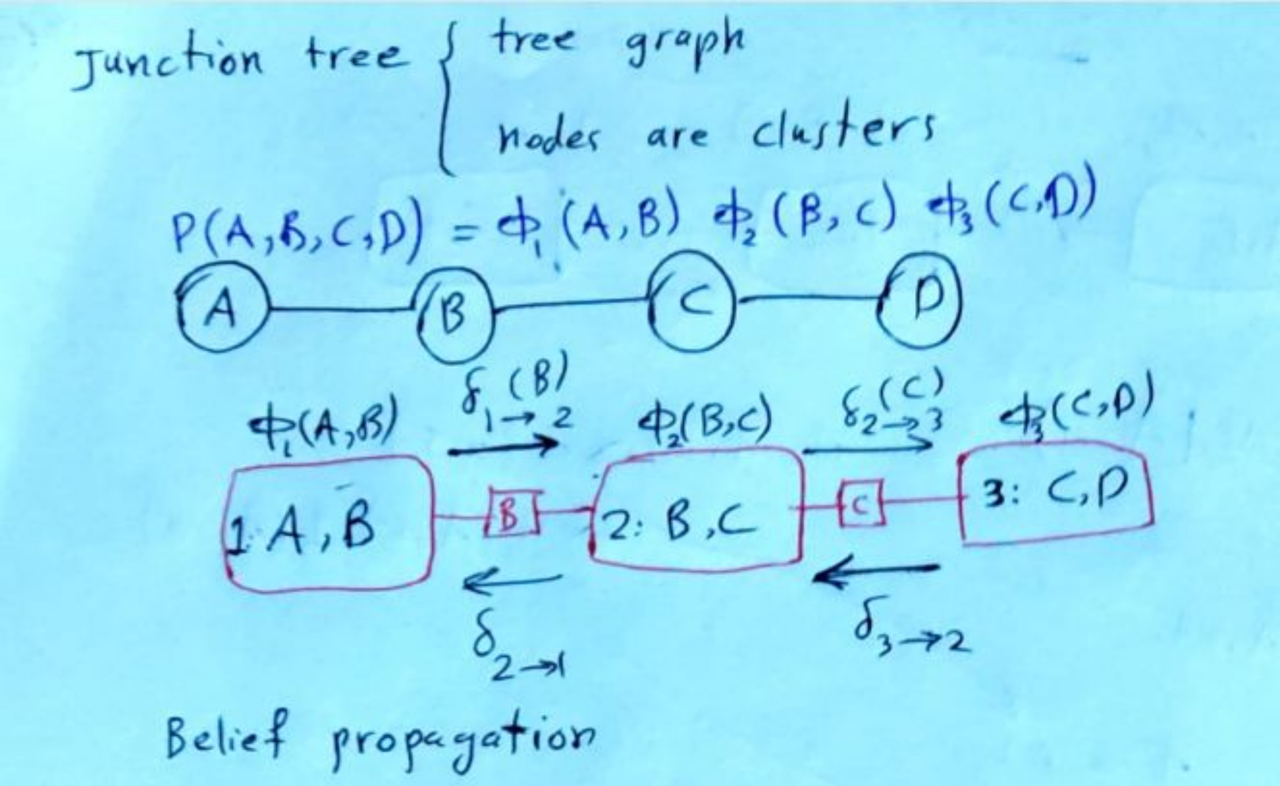
at the begining we can only pass
1 → 2
6 → 4
5 → 4
3 → 2

A message from cluster i to cluster j can only be sent if all incoming message to cluster i (except the one from cluster j) are sent (are ready).

⟹ the cluster graph must be a tree (Junction-tree algorithm → Exact Inference)

start from the leaves of the tree.

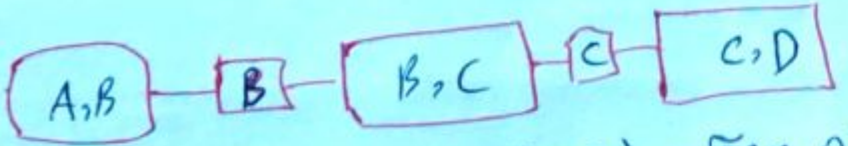
Junction Tree



inference with evidence

What if we have evidence? $P(X_t \mid X_e = x_e)$

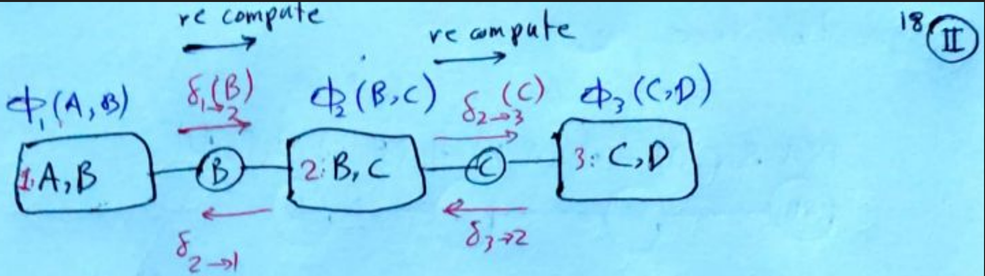
case 1: if X_t, X_e are in the same cluster



$P(D \mid C=c) \Rightarrow ?$ compute $\beta(C,D) = \tilde{P}(C,D) = Z\, P(C,D)$

$\tilde{P}(D, C=c) = \beta(D,c)$ $\qquad P(D \mid C=c) = \dfrac{\beta(D,c)}{\sum_{D'} \beta(D',c)} = \dfrac{\tilde{P}(D,c)}{\sum_{D} \tilde{P}(D,c)}$

inference with evidence



$P(C,D \mid A=a)$ $\qquad \phi_1(A,B) \to \phi_1(a,B)$

$\qquad\qquad \to \boxed{\phi_1(A,B)\ \mathbb{1}(A=a)}$

old

$\delta_{1\to 2}(B) = \sum_A \phi_1(A,B)$

new $\delta_{1\to 2}(B) = \phi_1(a,B)$

$= \sum_A \phi_1(A,B)\ \mathbb{1}(A=a) = \cancel{\Sigma}\ \phi_1(a,B)$

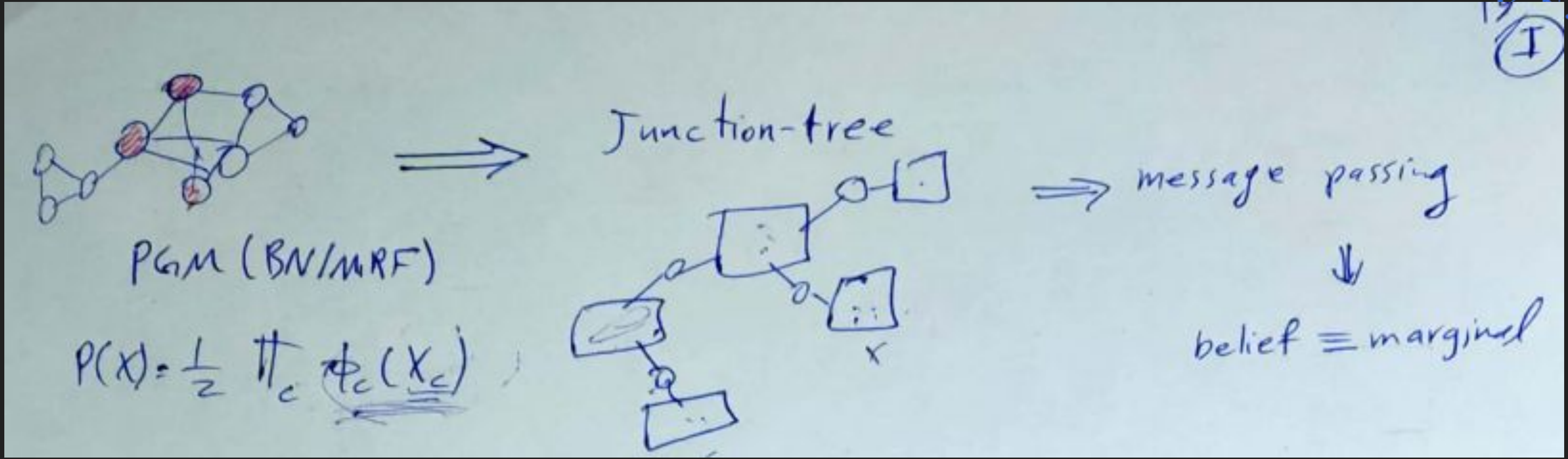
only need to recompute $\delta_{1\to 2}, \delta_{2\to 3}$

$P(B,C \mid A=a) = \checkmark \Rightarrow$ only need to recompute $\delta_{1\to 2}(B)$

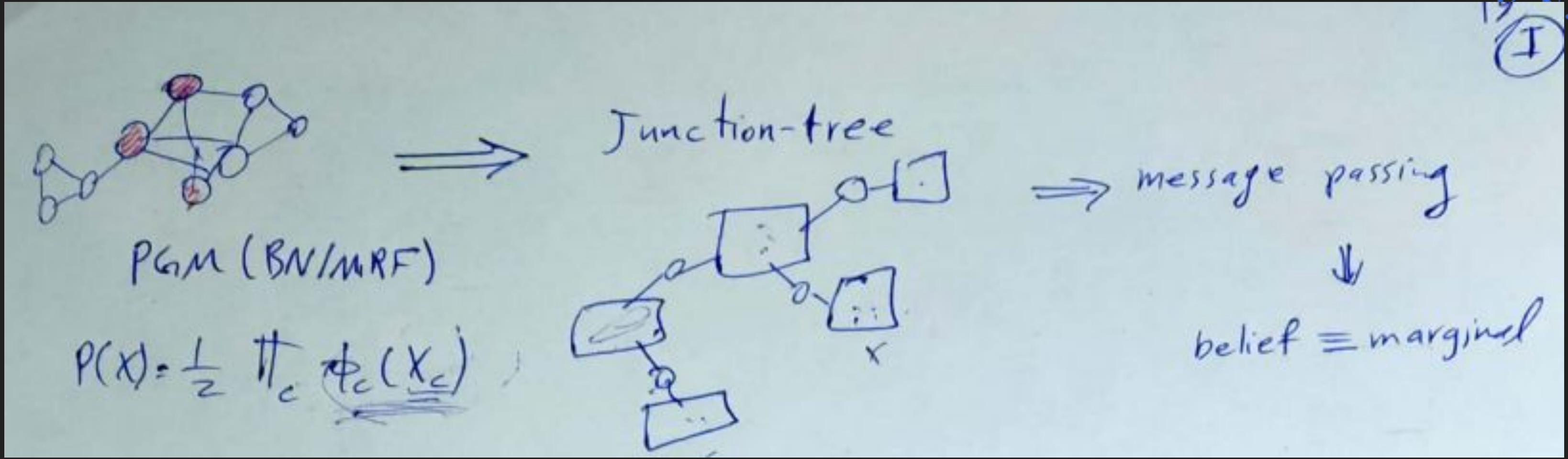
Junction tree

Message Passing is applicable only when the clusters are small (each cluster has few variables)

How to build a cluster tree?



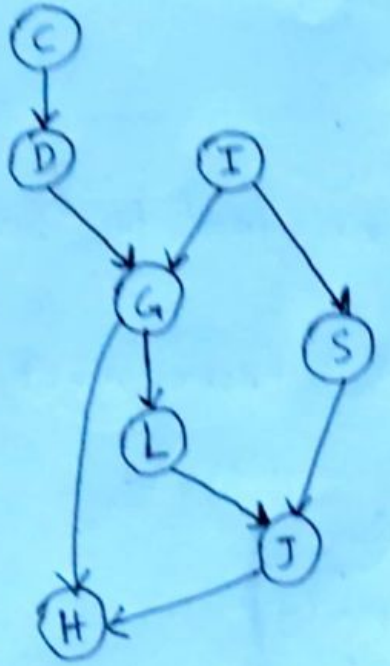
How to build a cluster tree?



1. Using variable elimination

Build cluster tree using VE

18 III

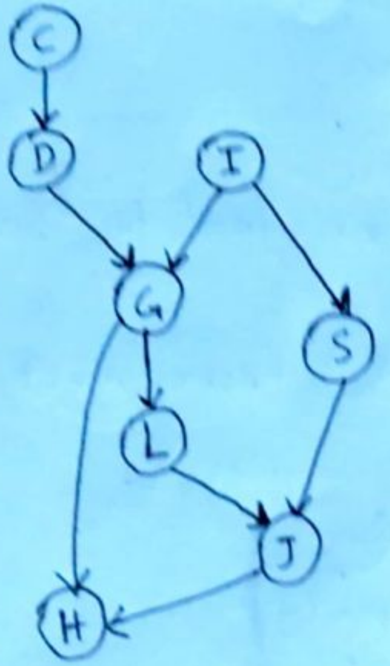


$$P(C,D,I,G,S,L,J,H)$$
$$= P(C)\,P(D|C)\,P(I)\,P(G|D,I)\,P(S|I)\,P(L|G)\,P(J|L,S)\,P(H|G,J)$$
$$= \phi_1(C)\,\phi_2(D,C)\,\phi_3(I)\,\phi_4(G,D,I)\,\phi_5(S,I)\,\phi_6(L,G)\,\phi_7(J,L,S)\,\phi_8(H,G,J)$$

Eliminate: C, D, I, H, G, S, L

Build cluster tree using VE

18 III



$$P(C, D, I, G, S, L, J, H)$$
$$= P(C)\, P(D \mid C)\, P(I)\, P(G \mid D, I)\, P(S \mid I)\, P(L \mid G)\, P(J \mid L, S)\, P(H \mid G, J)$$
$$= \phi_1(C)\, \phi_2(D, C)\, \phi_3(I)\, \phi_4(G, D, I)\, \phi_5(S, I)\, \phi_6(L, G)\, \phi_7(J, L, S)\, \phi_8(H, G, J)$$

Eliminate: C, D, I, H, G, S, L

How to build a cluster tree?

1. Using variable elimination
2. Determine valid cluster trees

How to build a cluster tree?

K. N. Toosi
University of Technology

Must possess tw properties

1- Family Preservation

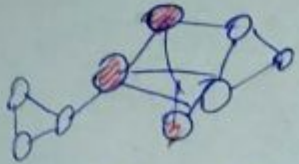
Each Factor can be assigned to some cluster.

For each factor $\phi_c(X_c)$ there must be a cluster C_i with variables X_{C_i} such that $X_c \subseteq X_{C_i}$

2- Running Intersection Property

For each variable X the subgraph ~~containg~~ of the cluster-tree containing X ~~form~~ is connected ($\equiv$ also forms a tree).

How to build a cluster tree?

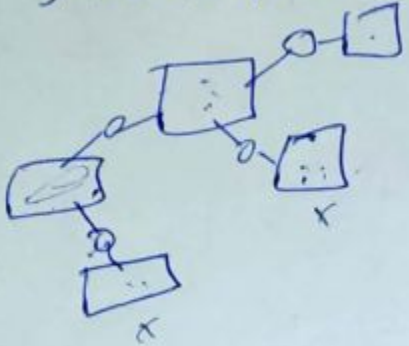


PGM (BN/MRF) $\Longrightarrow$ Junction-tree $\Longrightarrow$ message passing

$\Downarrow$

belief $\equiv$ marginal

$$P(X) = \frac{1}{Z} \prod_c \phi_c(X_c)$$



1- Family Preservation Property

2- Running Intersection Property (RIP)